1.1. Fourier Transform on R^d. $\mathcal{S}$ is the space of functions f such that f and all its derivatives of all orders decay faster than any inverse power at infinity. That is, $f \in \mathcal{S}$, if for any non negative integers $k, k_1, \cdots, k_d$, there is a constant $C(k_1, \ldots, k_d, k)$, such that

$$|\frac{\partial^{k_1+\cdots k_d} f}{\partial x_1^{k_1} \cdots \partial x_d^{k_d}}(x)| \leq C(k_1, \ldots, k_d, k)(1 + \|x\|)^{-k}$$

The Fourier transform of f is

$$g(y) = \widehat{f}(y) = (\frac{1}{\sqrt{2\pi}})^d \int_{R^d} f(x)\, e^{i\,\langle x,y\rangle} dx$$

The Fourier transform maps $\mathcal{S} \to \mathcal{S}$.

$$D_j \widehat{f} = \widehat{ix_j f}$$

$$y_j \widehat{f}(y) = -i\widehat{D_j f}$$

The Fourier transform of $f_a(x) = f(x-a)$ is

$$\begin{aligned}
\widehat{f_a}(y) &= (\frac{1}{\sqrt{2\pi}})^d \int_{R^d} f(x-a)\, e^{i\,\langle x,y\rangle} dx \\
&= (\frac{1}{\sqrt{2\pi}})^d \int_{R^d} f(x)\, e^{i\,\langle x+a,y\rangle} dx \\
&= (\frac{1}{\sqrt{2\pi}})^d \int_{R^d} f(x)\, e^{i\,\langle x,y\rangle}\ e^{i\,\langle a,y\rangle} dx \\
&= e^{i\,\langle a,y\rangle} \widehat{f}(y)
\end{aligned}$$

The Fourier transform of the convolution

$$g(x) = (f_1 * f_2)(x) = \int_{R^d} f_1(x-z) f_2(z) dz = \int_{R^d} f_1(z) f_2(x-z) dz$$

is

$$\widehat{g}(y) = \int f_1(x-z) f_2(z) e^{i\,\langle x,y\rangle} dz dx = \int f_1(x) f_2(z) e^{i\,\langle x+z,y\rangle} dx dz = \widehat{f_1}(y)\widehat{f_2}(y)$$

$$\int_{R^2} e^{-\frac{\|x\|^2}{2}} dx = \int_0^\infty \int_0^{2\pi} r e^{-\frac{r^2}{2}} dr d\theta = 2\pi$$

$$(\frac{1}{\sqrt{2\pi}})^d \int_{R^d} e^{-\frac{\|x\|^2}{2}} e^{\langle\theta,x\rangle} dx = e^{\frac{\|\theta\|^2}{2}}$$

By analytic continuation

$$(\frac{1}{\sqrt{2\pi}})^d \int_{R^d} e^{-\frac{\|x\|^2}{2}} e^{i\langle x,y\rangle} = e^{-\frac{\|y\|^2}{2}}$$

$$(\frac{1}{\sqrt{2\pi\epsilon}})^d \int_{R^d} e^{-\frac{\|x\|^2}{2\epsilon}} e^{i\langle x,y\rangle} = e^{-\frac{\epsilon\|y\|^2}{2}}$$

$$(\frac{\sqrt{\epsilon}}{\sqrt{2\pi}})^d \int_{R^d} e^{-\frac{\epsilon}{2}\|x\|^2} e^{i\langle x,y\rangle} = e^{-\frac{\|y\|^2}{2\epsilon}}$$

Theorem 1.1. If $f \in \mathcal{S}$ so is $\widehat{f}$. More over

$$f(x) = (\frac{1}{\sqrt{2\pi}})^d \int_{R^d} \widehat{f}(y)\, e^{-i\,\langle x,y\rangle} dy$$

Proof: Let $f_\epsilon(x) = (f * \phi_\epsilon)(x)$ where $\phi_\epsilon = (\frac{1}{\sqrt{2\pi\epsilon}})^d e^{-\frac{1}{2\epsilon}\|x\|^2}$.

$$\widehat{f_\epsilon}(y) = \widehat{f}(y) e^{-\frac{\epsilon}{2}\|y\|^2}$$

$$\begin{aligned}
(\frac{1}{\sqrt{2\pi}})^d \int_{R^d} \widehat{f_\epsilon}(y) e^{-i\langle x,y\rangle} dy &= (\frac{1}{\sqrt{2\pi}})^d \int_{R^d} \widehat{f}(y) e^{-\frac{\epsilon}{2}\|y\|^2} e^{-i\langle x,y\rangle} dy \\
&= (\frac{1}{2\pi})^d \int_{R^d}\int_{R^d} f(z) e^{-\frac{\epsilon}{2}\|y\|^2} e^{-i\langle x,y\rangle + i\,\langle z,y\rangle} dydz \\
&= \int_{R^d} (\frac{1}{\sqrt{2\pi\epsilon}})^d f(z) e^{-\frac{1}{2\epsilon}(x-z)^2} dz \\
&= \int_{R^d} f(z) K_\epsilon(x-z) dz
\end{aligned}$$

Let $\epsilon \to 0$. Dominated convergence theorem on the left and approximation of identity on the right.

$$(\frac{1}{\sqrt{2\pi\epsilon}})^d \int_{R^d} \widehat{f}(y) e^{-i\langle x,y\rangle} dy = f(x)$$

Theorem 1.2. For $f \in \mathcal{S}$

$$\int_{R^d} |f(x)|^2 dx = \int_{R^d} |\widehat{f}(y)|^2 dy$$

Proof.

$$\begin{aligned}
\int_{R^d} |\widehat{f}(y)|^2 e^{-\frac{\epsilon}{2}\|y\|^2} dy &= (\frac{1}{2\pi})^d \int_{R^d} \left| \int_{R^d} f(x) e^{i\langle x,y\rangle} dx \right|^2 e^{-\frac{\epsilon}{2}\|y\|\|^2} dy \\
&= (\frac{1}{2\pi})^d \int_{R^d}\int_{R^d}\int_{R^d} f(x)\overline{f(z)} e^{i\langle x-z,y\rangle} e^{-\frac{\epsilon}{2}\|y\|\|^2} dxdzdy \\
&= \int_{R^d}\int_{R^d} f(x)\overline{f(z)} K_\epsilon(x-z) dxdz
\end{aligned}$$

Let $\epsilon \to 0$.

Hence the map $f \to \widehat{f}$ extends from $\mathcal{S} \to \mathcal{S}$ as an isomorphism from $L_2(R^d) \to L_2(R^d)$.

$$\langle f, g\rangle_{L_2(R^d)} = \langle \widehat{f}, \widehat{g}\rangle_{L_2(R^d)}$$

Theorem 1.3. For $f \in L_2(R^d)$

$$\lim_{\ell\to\infty} \int_{R^d} \left| \left(\frac{1}{\sqrt{2\pi}}\right)^d \int_{\max_i |x_i| \le \ell} f(x) e^{i\langle x,y\rangle} dx - \widehat{f}(y)\right|^2 dy = 0$$

Proof. Note that $\|f\mathbf{1}_{\max_i |x_i|\le \ell} - f\|_{L_2(R^d} \to 0$ as $\ell \to \infty$

Interpolation. The Fourier transform T maps $L_1(R^d) \to L_\infty(R^d)$ and $L_2(R^d) \to L_2(R^d)$. By interpolation it will map for $1 \le p \le 2$, $L_p(R^d) \to L_{\frac{p}{p-1}}(R^d)$

Problem. For $f \in L_p(R^d)$ with $1 \le p \le 2$ does $\lim_{\ell\to\infty} \int_{|x|\le \ell} f(x) e^{i\langle x,y\rangle} dx$ exists in $L_{\frac{p}{p-1}}(R^d)$? Give an example of a function f for which $\lim_{\ell\to\infty} \int_{|x|\le\ell} f(x) e^{ixy} dx$ does not exist in any L_p with $p \ge 1$. Can you find one in $L_3(R^d)$?

1.2. Positive definite functions.

Theorem 1.4. If μ is a finite nonnegative measure on R^d its Fourier transform $g(y) = \int_{R^d} e^{i\langle x,y\rangle} d\mu$ is a bounded continuous function on R^d. It is positive definite in the sense that for any finite collection $\{x_j\} \in R^d$ and complex numbers $\{z_j\} \in \mathbf{C}$

$$\sum_{i,j} g(y_i - y_j) z_i \overline{z}_j \ge 0$$

Conversely, if $g(y)$ is a continuous positive definite function on R^d, there is a unique nonnegative finite measure μ such that

$$g(y) = \int_{R^d} e^{i\langle x,y\rangle} d\mu$$

Proof.

$$\sum_{i,j} g(y_i - y_j) z_i \overline{z}_j = \int_{R^d} \left| \sum_j z_j e^{i\langle y_j, x\rangle} \right|^2 \mu(dx) \ge 0$$

As for the converse we remark that if g is a positive definite function so is $h(y) = g(y) e^{i\langle a,y\rangle}$ for any $a \in R^d$.

$$\sum_{i,j} h(y_i - y_j) z_i \bar{z}_j = \sum_{i,j} g(y_i - y_j) e^{i\langle a, y_i - y_j\rangle} z_i \bar{z}_j = \sum_{i,j} g(y_i - y_j)(e^{i\langle a,y_i\rangle} z_i)\overline{(e^{i\langle a,y_j\rangle} z_j)} \ge 0$$

Convex combinations and constant multiples of positive definite functions are positive definite. Limits of positive definite functions are positive definite. If g is positive definite so is $\int_{R^d} g(y)e^{i\langle x,y\rangle}\phi(x)dx$ for $\phi \geq 0$. Taking $\phi(x) = (\frac{1}{\sqrt{2\pi\epsilon}})^d e^{-\frac{1}{2\epsilon}\|x\|^2}$ we get that $g(y)e^{-\frac{\epsilon}{2}\|y\|^2}$ is again positive definite. It is therefore sufficient to note that if $g(y)$ is positive definite, continuous and integrable, then

$$\int_{R^d} g(y)e^{-\frac{\epsilon}{2}\|y\|^2}dy = (\frac{\sqrt{\epsilon}}{\sqrt{2\pi}})^d \int_{R^d}\int_{R^d} g(y-z)e^{-\epsilon\|y\|^2}e^{-\epsilon\|z\|^2}dydz \geq 0$$

Hence $f_\epsilon(x) = \frac{1}{\sqrt{2\pi}}\int_{R^d} g(y)e^{-\frac{\epsilon}{2}\|y\|^2}e^{i\langle x,y\rangle}dy \geq 0$ and $\int_{R^d} f_\epsilon(x)dx = \sqrt{2\pi}g(0)$. Then $f_\epsilon(x)dx$ has a limit as a nonnegative finite measure.

We will meed the following facts. If $g(y)$ is continuous and positive definite then $g(0) \geq 0$ and $|g(y)| \leq g(0)$. More over

$$|g(y_1) - g(y_2)| \leq o(|y_1 - y_2|)$$

To see this the matrix $\begin{pmatrix} g(0) & g(y) \\ g(-y) & g(0) \end{pmatrix}$ is positive definite. Forces $g(-y) = \overline{g(y)}$ and $|g(y)|^2 \leq [g(0)]^2$. Similarly the positive definiteness of the matrix

$$\begin{pmatrix} g(0) & g(a) & g(b) \\ g(-a) & g(0) & g(a-b) \\ g(-b) & g(b-a) & g(0) \end{pmatrix}$$

yields the inequality $|g(a) - g(b)|^2 \leq 1 - |g(a-b)|^2$. Finally if $g(y) = \int e^{i\langle x,y\rangle}d\mu$ where μ is a nonnegative measure on R^d,

$$(2\epsilon)^{-d}\int_{\{y:\sup_i |y_i|\leq\epsilon\}}[g(0) - g(y)]dy = \int [1 - \Pi(\frac{\sin \epsilon x_i}{\epsilon x_i})]\mu(dx) \geq \frac{1}{2}\mu[x : \sup_i |x_i| \geq 2\epsilon^{-1}]$$

This estimate allows us to choose a subsequence that has a limit μ in the weak topology which will be a probability distribution with g as its characteristic function.